Brownian motion and Stochastic Calculus
Dylan Possamaï

Assignment 12—solutions

Exercise 1

Let $(B_t)_{t\in[0,T]}$ be a Brownian motion in $[0,T]$ and a_1, a_2, b_1, b_2 deterministic functions of time. The general form of a scalar *linear stochastic differential equation* is

$$\mathrm{d}X_t = \big(a_1(t)X_t + a_2(t)\big)\mathrm{d}t + \big(b_1(t)X_t + b_2(t)\big)\mathrm{d}B_t. \tag{0.1}$$

If the coefficients are measurable and bounded on $[0,T]$, we can apply our general result to get existence and uniqueness of a strong solution $(X_t)_{t\in[0,T]}$ for each initial condition x.

1) When $a_2(t)\equiv 0$ and $b_2(t)\equiv 0$, (0.1) reduces to the *homogeneous linear SDE*

$$\mathrm{d}X_t = a_1(t)X_t\mathrm{d}t + b_1(t)X_t\mathrm{d}B_t. \tag{0.2}$$

Show that the solution of (0.2) with initial data $x=1$ is given by

$$X_t = \exp\bigg(\int_0^t \Big(a_1(s) - \frac{1}{2}b_1^2(s)\Big)\mathrm{d}s + \int_0^t b_1(s)\mathrm{d}B_s\bigg).$$

2) Find a solution of the SDE (0.1) with initial condition $X_0 = x$.

3) Solve the *Langevin's SDE*

$$\mathrm{d}X_t = a(t)X_t\mathrm{d}t + \mathrm{d}B_t,\ X_0 = x.$$

1) **Write** $X_t = \mathrm{e}^{V_t}$ **with** $V_t = \int_0^t \big(a_1(s) - \frac{1}{2}b_1^2(s)\big)\mathrm{d}s + \int_0^t b_1(s)\mathrm{d}B_s$. **Then**

$$\mathrm{d}X_t = \mathrm{e}^{V_t}\mathrm{d}V_t + \frac{1}{2}\mathrm{e}^{V_t}\mathrm{d}[V]_t.$$

Plug the expression for V_t

$$\begin{aligned}\mathrm{d}X_t &= \mathrm{e}^{V_t}\bigg(\Big(a_1(t) - \frac{1}{2}b_1^2(t)\Big)\mathrm{d}t + b_1(t)\mathrm{d}B_t\bigg) + \frac{1}{2}\mathrm{e}^{V_t}b_1^2(t)\mathrm{d}t \\ &= X_t\big((a_1(t)\mathrm{d}t + b_1(t)\mathrm{d}B_t\big).\end{aligned}$$

2) **Consider a process** $(U_t)_{t\geq 0}$ **given by the solution of an homogeneous linear SDE, which by** 1)**, is given in explicit form by**

$$U_t = \exp\bigg(\int_0^t \Big(a_1(s) - \frac{1}{2}b_1^2(s)\Big)\mathrm{d}s + \int_0^t b_1(s)\mathrm{d}B_s\bigg).$$

Now we want to find the coefficients $a_2(t)$ **and** $b_2(t)$ **such that** $X_t = U_tV_t$, **where**

$$\mathrm{d}V_t = \alpha(t)\mathrm{d}t + \beta(t)\mathrm{d}B_t,$$

for appropriate maps α **and** β. **Applying Itô's formula**

$$\beta(t)U_t = b_2(t),\ \textbf{and}\ \alpha(t)U_t = \alpha(t) - b_1(t)b_2(t).$$

To sum up

$$X_t = U_t\bigg(x + \int_0^t \frac{a_2(s) - b_1(s)b_2(s)}{U_s}\mathrm{d}s + \int_0^t \frac{b_2(s)}{U_s}\mathrm{d}B_s\bigg).$$

3) **Applying** 2) **with** $U_t = \exp\big(\int_0^t a(s)\mathrm{d}s\big)$ **we find**

$$X_t = \exp\bigg(\int_0^t a(s)\mathrm{d}s\bigg)\bigg(X_0 + \int_0^t \exp\bigg(-\int_0^u a(s)\mathrm{d}s\bigg)\mathrm{d}B_u\bigg).$$

Exercise 2

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space carrying a one-dimensional Brownian motion B, whose $\mathbb{P}$-augmented filtration is denoted by $\mathbb{F}$. Fix positive constants T and γ, and let ξ be a bounded $\mathcal{F}_T$-measurable random variable.

1) Show that the process
$$Y_t := -\gamma \log\Big(\mathbb{E}^{\mathbb{P}}\big[\mathrm{e}^{-\xi/\gamma}\big|\mathcal{F}_t\big]\Big),\ t \geq 0,$$
is the first component of a solution to the BSDE with terminal condition ξ (at T) and generator g with
$$g(z) := -\frac{1}{2\gamma}z^2,\ z \in \mathbb{R}.$$

2) Let $b \in \mathbb{R}$. Show that the process
$$Y_t := -\gamma\bigg(\frac{b^2}{2}(T-t) - bB_t + \log\Big(\mathbb{E}^{\mathbb{P}}\big[\mathrm{e}^{bB_T-\xi/\gamma}\big|\mathcal{F}_t\big]\Big)\bigg),\ t \geq 0,$$
is the first component of a solution to the BSDE with terminal condition ξ (at T) and generator g with
$$g(z) := -\frac{1}{2\gamma}z^2 - bz,\ z \in \mathbb{R}.$$

1) **Let** $P := \mathrm{e}^{-Y/\gamma}$**. It is immediate to check that** P **is a (bounded) martingale in the Brownian filtration, so that we can write using the martingale representation theorem**

$$\mathrm{d}P_t = Q_t \mathrm{d}B_t,$$

for some $\mathbb{Q} \in \mathbb{H}^2(\mathbb{R}, \mathbb{F}, \mathbb{P})$**. Then, applying Itô's formula to** $f(P_t)$ **where** $f(x) := -\gamma\log(x)$ **(recall that** P **is positive by definition) gives the desired result.**

2) **The reasoning is the same: use** 1) **to get the dynamics for** $-\gamma\log\big(\mathbb{E}^{\mathbb{P}}\big[\mathrm{e}^{bB_T-\xi/\gamma}\big|\mathcal{F}_t\big]\big)$**, and then apply Itô to get the dynamics of** Y**.**

Exercise 3

Let $(B_t)_{t\geq 0}$ be a Brownian motion defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(X_t)_{t\geq 0}$ the unique solution of the SDE

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + g(X_t)\mathrm{d}B_t,\ X_0 = x,$$

where $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ are Lipschitz-continuous functions.

1) Find a non-constant function $\phi(x) \in C^2(\mathbb{R}, \mathbb{R})$ such that $Y_t := \phi(X_t)$ is a local martingale. Moreover, derive a SDE for $(Y_t)_{t\geq 0}$.

   Hint: Prove and use that general solution of the ODE: $y'f(x) + \frac{1}{2}y''g^2(x) = 0$ is of the form
   $$y(x) = a + b\int_0^x \exp\bigg(-2\int_0^u \frac{f(v)}{g^2(v)}\mathrm{d}v\bigg)\mathrm{d}u,\ (a,b) \in \mathbb{R}^2.$$

2) Assume additionally that f is negative on $(-\infty, 0)$ and positive on $[0, \infty)$. Show that in this case, Y is a martingale.

1) **Applying Itô's formula, we obtain that $\mathbb{P}$–a.s., for all $t \geq 0$**

$$Y_t = \phi(X_t) = \phi(x) + \int_0^t \phi'(X_s)g(X_s)\mathrm{d}B_s + \int_0^t \Big(\phi'(X_s)f(X_s) + \frac{1}{2}\phi''(X_s)g^2(X_s)\Big)\mathrm{d}s.$$

Thus, we obtain that Y is a local martingale if and only if for any $x \in \mathbb{R}$, $\phi(x)$ satisfies the following ordinary differential equation

$$\phi'(x)f(x) + \frac{1}{2}\phi''(x)g^2(x) = 0.$$

It is easy to check by direct integration that the general solution of the above ordinary differential is of the form

$$\phi(x) = a + b\int_0^x \exp\Big(-2\int_0^u \frac{f(v)}{g^2(v)}\mathrm{d}v\Big)\mathrm{d}u,\ (a,b) \in \mathbb{R}^2. \tag{0.3}$$

For the second part, let $\phi(x)$ be of the form (0.3) with $b \neq 0$ (i.e. a non trivial solution). We first observe that ϕ is continuous and increasing, hence the inverse function of ϕ, denoted by $\phi^{(-1)}$, exists. From 1), we know that $\mathbb{P}$–a.s. for any $t \geq 0$

$$Y_t = \phi(X_t) = \phi(x) + \int_0^t \phi'(X_s)g(X_s)\mathrm{d}B_s. \tag{0.4}$$

Thus, as $X_t = \phi^{-1}(Y_t)$, we get that Y_t satisfies the SDE

$$\mathrm{d}Y_t = (\phi' \circ \phi^{-1})(Y_t)(g \circ \phi^{-1})(Y_t)\mathrm{d}B_t,\ Y_0 = \phi(x).$$

2) **As $\phi(X)$ is a continuous local martingale of the form (0.4), it is enough to check that for any $T > 0$**

$$\mathbb{E}^{\mathbb{P}}\Big[\int_0^T \big(\phi'(X_s)g(X_s)\big)^2\mathrm{d}s\Big] < \infty,$$

to conclude that $(\phi(X_t))_{t\geq 0}$ is a true martingale. First, we observe that due to our additional assumption on f being negative on $(-\infty, 0]$ and positive on $(0,\infty)$, we obtain that

$$\sup_{x\in\mathbb{R}} \big|\phi'(x)\big| \leq |b|.$$

Moreover, as $g : \mathbb{R} \longrightarrow \mathbb{R}$ is Lipschitz-continuous, there exists a constant $k > 0$ such that

$$|g(x)| \leq |g(0)| + k|x|.$$

As for any $(a,b) \in \mathbb{R}^2$, we have $(a+b)^2 \leq 2(a^2+b^2)$, we obtain that

$$g(x)^2 \leq 2g(0)^2 + 2k^2x^2.$$

We conclude that there are constants $C, D > 0$ such that

$$\mathbb{E}^{\mathbb{P}}\Big[\int_0^T \big(\phi'(X_s)g(X_s)\big)^2\mathrm{d}s\Big] \leq C + D\mathbb{E}^{\mathbb{P}}\Big[\int_0^T X_s^2\mathrm{d}s\Big].$$

But this is finite as $(X_t)_{t\geq 0}$ is by assumption the strong solution of the SDE

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + g(X_t)\mathrm{d}B_t,\ X_0 = x,$$

and therefore, for any $T > 0$

$$\mathbb{E}^{\mathbb{P}}\Big[\sup_{0\leq t\leq T} |X_t|^2\Big] = \mathbb{E}^{\mathbb{P}}\Big[\sup_{0\leq t\leq T} |X_t|^2\Big] < \infty.$$

Exercise 4

1) Let $(f_t)_{t\geq 0}$ be an $\mathbb{F}$-adapted, positive, increasing, differentiable process starting from zero and consider the following SDE

$$\mathrm{d}X_t = \sqrt{f_t'}\mathrm{d}B_t. \tag{0.5}$$

Show that the process B_{f_t} is a weak solution of (0.5).

Hint: in other words, given a Brownian motion $(B_t)_{t\geq 0}$ and a function f satisfying the previous assumptions, there exists a Brownian motion $(\widehat{B}_t)_{t\geq 0}$, such that

$$\mathrm{d}\widehat{B}_{f_t} = \sqrt{f_t'}\mathrm{d}B_t.$$

2) Recall that a solution of the SDE

$$\mathrm{d}X_t = -\gamma X_t \mathrm{d}t + \sigma \mathrm{d}B_t,\ X_0 = x, \tag{0.6}$$

is called Ornstein–Uhlenbeck process. Show that an Ornstein–Uhlenbeck process has representation

$$X_t = \mathrm{e}^{-\gamma t}\widetilde{B}_{\psi(t)},$$

where

$$\psi(t) := \frac{\sigma^2(\mathrm{e}^{2\gamma t}-1)}{2\gamma},$$

and where $(\widetilde{B}_t)_{t\geq 0}$ is a Brownian motion started at x.

3) Consider the SDE

$$\mathrm{d}X_t = \sigma(X_t)\mathrm{d}B_t,\ X_0 = x, \tag{0.7}$$

with $\sigma(x) > 0$ such that

$$G(t) := \int_0^t \frac{\mathrm{d}s}{\sigma^2(B_s)},$$

is finite for finite t, and increases to infinity, that is $G(\infty) = \infty$, $\mathbb{P}$–a.s. Under these assumptions, the inverse of G is well-defined, and we let

$$\tau_t := G_t^{(-1)}.$$

Show that the process $X_t := B_{\tau_t}$ is a weak solution to the SDE (0.7).

1) **Given the assumptions made on f, it admits an inverse g. Let then**

$$\widehat{B}_t := \int_0^{g_t} \sqrt{f_s'}\mathrm{d}B_s.$$

By definition, we have that $\widehat{B}_{f_t}$ satisfies the required equation, so we just need to check that $\widehat{B}$ is a Brownian motion. It is direct to check that this is a continuous local martingale and that

$$\big[\widehat{B}\big]_t = \int_0^{g_t} f_s'\mathrm{d}s = (f\circ g)(t) - (f\circ g)(0) = t,$$

and we can conclude by Lévy's characterisation.

2) **Both the Ornstein–Uhlenbeck process and X as defined in the question are continuous, centred Gaussian processes. it thus suffices to compute the covariance function to make sure that their distributions match. Recall that if $\widetilde{X}$ is the Ornstein–Uhlenbeck process, we have for $0\leq s\leq t$**

$$\mathbb{C}\mathrm{ov}^{\mathbb{P}}\big[\widetilde{X}_t,\widetilde{X}_s\big] = \mathbb{C}\mathrm{ov}^{\mathbb{P}}\bigg[\mathrm{e}^{-\gamma t}\int_0^t \sigma\mathrm{e}^{\gamma u}\mathrm{d}B_u, \mathrm{e}^{-\gamma s}\int_0^s \sigma\mathrm{e}^{\gamma u}\mathrm{d}B_u\bigg] = \mathrm{e}^{-\gamma(t+s)}\int_0^{t\wedge s}\sigma^2\mathrm{e}^{2\gamma u}\mathrm{d}u = \mathrm{e}^{-\gamma(t+s)}\psi(s).$$

Then, using the covariance function for Brownian motion and the fact that ψ is non-decreasing

$$\mathbb{C}\mathrm{ov}^{\mathbb{P}}[X_t,X_s] = \mathrm{e}^{-\gamma(t+s)}\mathbb{C}\mathrm{ov}^{\mathbb{P}}\big[\widetilde{B}_{\psi(t)},\widetilde{B}_{\psi(s)}\big] = \mathrm{e}^{-\gamma(t+s)}\psi(s),$$

hence the result.

3) **The operator associated to the SDE is given by**

$$Lf(x) = \frac{1}{2}\sigma^2(x)f''(x).$$

We want to show that $X_t = B(\tau_t)$ **is a solution to the martingale problem for** L**. Take** $f \in C_0^2$, **then we know that the process**

$$M_t := f(B_t) - \int_0^t \frac{1}{2}f''(B_s)\mathrm{d}s,$$

is a martingale. Moreover $(\tau_t)_{t\geq 0}$ **is an increasing sequence of stopping times and so the process** M_{τ_t} **is a martingale. Now we want the find an explicit expression for the process** τ. **Using the formula for the derivative of the inverse function,**

$$(G^{(-1)})'_t = \frac{1}{G'(G_t^{(-1)})} = \frac{1}{\sigma^2(B(G^{(-1)})_t)} = \sigma^{-2}(B_{\tau_t}). \tag{0.8}$$

From (0.8) **we see that** $(\tau_t)_{t\geq 0}$ **satisfies** $d\tau_t = \sigma^{-2}(B_{\tau_t})\mathrm{d}t$. **Now perform a change of variable** $s = \tau_u$ **to obtain that the process**

$$f(B_{\tau_t}) - \int_0^t \frac{1}{2}\sigma^2(B_{\tau_u})f''(X_u)\mathrm{d}u,$$

is a martingale and so $(X_t)_{t\geq 0}$ **solves the martingale problem for** L**.**